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THEORY OF THE STEADY LAMINAR NATURAL CONVECTION ABOVE A HORIZONTAL LINE HEAT SOURCE AND A POINT HEAT SOURCE

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Abstract—Steady laminar natural convection above a horizontal line heat source and a point heat source are analysed mathematically. The solutions of elementary functions are given for $Pr = 2$, and also for the flow above a point source for $Pr = 1$.

The velocity and temperature distributions for the case of Prandtl number equal to 0·01, 0·7 and 10 are computed with an electronic computer, and differences, caused by different Prandtl number, among the velocity distributions or the temperature distributions are described in detail.

NOMENCLATURE

- c_p, specific heat of the fluid at constant pressure;
- f, nondimensional stream function, defined in equation (6) or (II, 6);
- g, gravitational acceleration;
- G, Grashof number defined in equation (7) or (II, 7);
- h, nondimensional temperature, defined in equation (6) or (II, 6);
- Pr, ν/κ, Prandtl number;
- Q_l, rate of heat transfer from unit length of a line source (kcal/mh);
- Q_p, rate of heat transfer from a point source (kcal/h);
- t, fluid temperature above the ambient temperature;
- u, v, velocity components indicated in Fig. 1;
- x, y, co-ordinate system indicated in Fig. 1;
- β, volumetric thermal expansion coefficient;
- γ, specific weight of the fluid (kg/m^3);
- Θ, number with dimension of temperature, defined in equation (13) or (II, 13);
- κ, $\lambda/c_p\gamma$, thermal diffusivity of the fluid;
- λ, thermal conductivity of the fluid;
- ν, kinematic viscosity of the fluid;
- ξ, independent variable of the "similar" solution, defined in equation (6) or (II, 6);
- ψ, stream function defined in equation (5) or (II, 5).

Subscripts

- l, for the line heat source;
- p, for the point heat source.

INTRODUCTION

IN NATURAL convective heat transfer, fluid particles in the boundary layer, after separating from the vicinity of the heating surface, also form a certain layer which is similar to the boundary layer. For example, a plume of heated fluid above a horizontal pipe or a column above a small body is formed. Though these circumstances are similar to the trailing vortex-sheets in the case of forced convection, buoyancy force exerts into the plume or the column itself in the case of natural convection. It therefore gives rise to a question whether or not the plume or the column has effects on the flow in the vicinity of the heating surface and hence on the heat transfer. Accordingly, we have taken up the question to clarify the fluid above a horizontal line heat source and that above a point heat source.

Formerly, Schuh [1] solved the same problems of natural convection in air. Recently, Spalding and Cruddace [2] calculated the velocity distribution of laminar buoyant flow above a line heat source in a fluid of large Prandtl number numerically. It seems that they studied it as a problem to heat heavy oil stored in a tank by horizontal steam pipes.

I. A HORIZONTAL LINE HEAT SOURCE

I. 1. *Basic equations*

For condition of two-dimensional convection above a line heat source located horizontally in large fluid space, the primary variables describing the flow and the temperature are taken as indicated in Fig. 1, where x and y are the vertical and horizontal co-ordinates with the origin at the heat source, and u, v are the velocity components of x, y direction respectively.

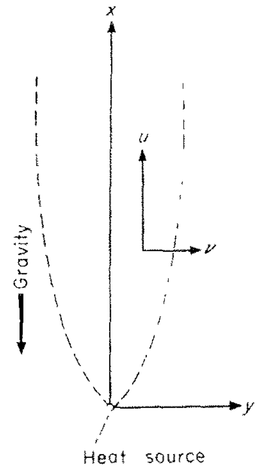

FIG. 1. Definition sketch.

When a plume is formed above the source, the terms of higher order in the equations of continuity, motion and energy may be neglected. Then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta t + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \kappa\frac{\partial^2 t}{\partial y^2}, \tag{3}$$

where g; gravitational acceleration, β; volumetric thermal expansion coefficient, t; fluid temperature above the ambient temperature, ν; kinematic viscosity of the fluid, κ; thermal diffusivity of the fluid. The velocity and temperature distributions must be symmetrical with x-axis, and the temperature and the vertical velocity component at a distance from the heat source are not to be affected by it. Hence the boundary conditions may be expressed as:

$$\begin{aligned} y &= 0;\ v = 0,\ \partial u/\partial y = 0,\ \partial t/\partial y = 0, \\ y &= \infty;\ u = 0,\ t = 0. \end{aligned} \tag{4}$$

When we introduce the stream function ψ such that

$$\frac{\partial\psi}{\partial y} = u, \quad \frac{\partial\psi}{\partial x} = -v, \tag{5}$$

and assume the existence of a "similar" solution of the field of the velocity and the temperature within the plume such that

$$\xi = G_l^{\frac{1}{5}}\frac{y}{x}, \ \psi = \nu G_l^{\frac{1}{5}} f(\xi), \quad t = G_l^{-\frac{1}{5}}\Theta_l h(\xi), \tag{6}$$

where

$$G_l = \frac{x^3 g\beta\Theta_l}{\nu^2} \tag{7}$$

and when Θ_l is reckoned as an arbitrary constant with dimension of temperature, we obtain the following ordinary differential equations:

$$\left.\begin{aligned} f''' + \tfrac{3}{5}ff'' - \tfrac{1}{5}f'^2 + h = 0, \\ h'' + \tfrac{3}{5}Pr(fh)' = 0, \end{aligned}\right\} \tag{8a, b}$$

with the boundary conditions:

$$\left.\begin{aligned} \xi = 0;\ f = 0,\ f'' = 0,\ h' = 0, \\ \xi = \infty;\ f' = 0,\ h = 0, \end{aligned}\right\} \tag{9a, b}$$

where $Pr = \nu/\kappa$ is Prandtl number and the prime denotes differentiation with respect to ξ. The vertical and lateral velocity components u and v are written as:

$$\frac{ux}{\nu} = G_l^{\frac{2}{5}} f', \ \frac{vy}{\nu} = -\xi(\tfrac{3}{5}f - \tfrac{2}{5}\xi f'). \tag{10}$$

In this phenomena, conservation of energy and that of momentum are respectively expressed as:

$$c_p\gamma\int_{-\infty}^{\infty} ut\,\mathrm{d}y = Q_l, \tag{11}$$

$$\int_{-\infty}^{\infty} u^2\,\mathrm{d}y = g\beta\int_0^x\left(\int_{-\infty}^{\infty} t\,\mathrm{d}y\right)\mathrm{d}x. \tag{12}$$

where c_p is the specific heat of the fluid at contants pressure, γ, the specific weight of the fluid, Q_l, the rate of heat transfer from unit length of the line source (kcal/mh). These are transformed by (6) and (10) as:

$$\Theta_l = \frac{1}{\int_{-\infty}^{\infty} f'h \, d\xi} \cdot \frac{Q_l}{c_p \gamma \nu}, \tag{13}$$

$$\int_{-\infty}^{\infty} f'^2 \, d\xi = \tfrac{5}{4} \int_{-\infty}^{\infty} h \, d\xi. \tag{14}$$

Θ_l in (6) and (7) is defined by (13), and it is convenient to normalize the solutions of (8) f and h by the following formula:

$$\int_{-\infty}^{\infty} f'h \, d\xi = 1. \tag{15)*$$

I. 2. *Analytical solution of the ordinary differential equations for Pr = 2*

Now, the case where the integrand of equation (14) is zero identically, namely

$$f'^2 - \tfrac{5}{4} h \equiv 0, \tag{16}$$

is considered. Therefore (8a) becomes:

$$f''' + \tfrac{3}{5} ff'' + \tfrac{2}{5} f'^2 = 0. \tag{17}$$

When (17) is integrated twice with boundary conditions (9a),

$$f' + \tfrac{3}{10} f^2 = \alpha, \tag{18}$$

where α is a positive integral constant, for it means physically non-dimensional vertical velocity component at $\xi = 0$.

Hence the integral of (18) is

$$f = \left(\frac{10\alpha}{3}\right)^{1/2} \tanh \left(\frac{3\alpha}{10}\right)^{1/2} \xi. \tag{19}$$

h is obtained by substituting (19) into (16) such that:

$$h = \tfrac{4}{5} \alpha^2 \operatorname{sech}^4 \left(\frac{3\alpha}{10}\right)^{1/2} \xi, \tag{20}$$

and this satisfies also the boundary conditions (9).

f and h given by (19) and (20) satisfy (8b) only in the case where Prandtl number is equal to two. α is determined by the normalizing condition (15):

$$\alpha = \tfrac{1}{4} \left(\tfrac{15}{2}\right)^{\frac{3}{5}} \risingdotseq 0{\cdot}837. \tag{21}$$

The nondimensional vertical velocity component is written as:

$$f' = \alpha \operatorname{sech}^2 \left(\frac{3\alpha}{10}\right)^{1/2} \xi. \tag{22}$$

I. 3. *Numerical calculation*

Equations (8) are held invariable for the transformation by

$$\xi = a\bar{\xi}, f = b\bar{f}, h = c\bar{h}, \tag{23}$$

as far as the constants a, b and c have the following relations:

$$ab = 1 \quad \text{and} \quad b^2 = ac. \qquad (24$$

Since these constants can be taken as arbitrary combination, the solution of (8) subject to the conditions (9) is not uniquely determined. Consequently, in numerical calculation, either $\bar{f}'(0)$ or $\bar{h}(0)$, which may be regarded as eigenvalues, can be taken to be arbitrary, and the other is guessed by the method of trial and error.

By substituting (23) into (15) we obtain the following equation:

$$bcI = 1, \tag{25}$$

where

$$I = \int_{-\infty}^{\infty} \bar{f}'\bar{h} \, d\bar{\xi}.$$

When a pair of solutions $\bar{f}$ and $\bar{h}$ satisfying the conditions (9) is found out, the above integral I is calculated and then constants a, b and c can be solved simultaneously by (24) and (25). Consequently, the solution of (8) subject to the conditions (9) and (15) becomes:

$$\xi = I^{\frac{1}{4}}\bar{\xi}, f = I^{-\frac{1}{4}}\bar{f}, h = I^{-\frac{3}{4}}\bar{h}. \tag{26}$$

In Figs. 2a and 2b and Fig. 3 are shown the variations of the nondimensional vertical velocity

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* Equations (8) are identical to Schuh's equations, where the condition expressed by (15), which is assumed in his transformation, is not clearly stated. On the other hand, Spalding and Cruddace introduce the condition (15) instead of the first condition of equations (9a). This may be correct logically, since the solution of (8) is not uniquely determined without condition (15), but in this case it may be reasonable to omit the second condition of (9b), which is reduced from (8b), rather than the first condition of (9a). Practically, the theoretical or numerical calculations cannot be performed without this condition.

f' and the nondimensional temperature h with ξ. In these figures the curves for $Pr = 2$ show the analytical solution and those for $Pr = 0{\cdot}01$, $0{\cdot}7$ and 10 show the numerical solutions by an electronic digital computer. In digital computation (8) are approximated in the form of finite difference, from which are derived simultaneous equations of the first degree which are called three terms equation. Details of these methods

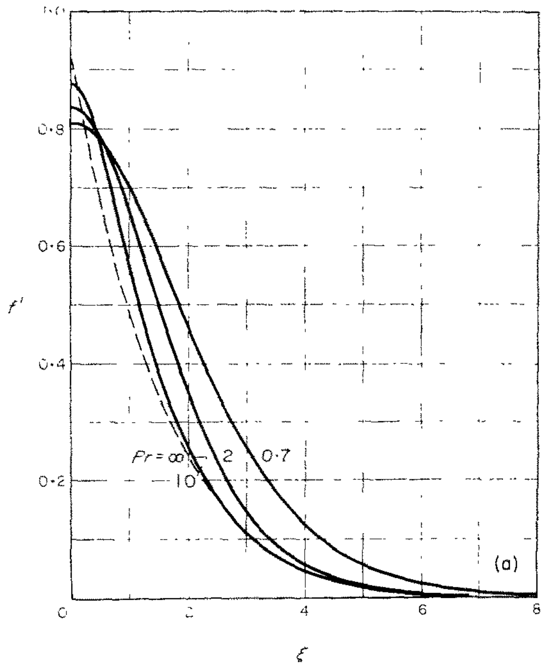

FIG. 2a. Nondimensional vertical velocity distributions above a line heat source, f'. The dotted line shows the Spalding–Cruddace's solution for $Pr \to \infty$.

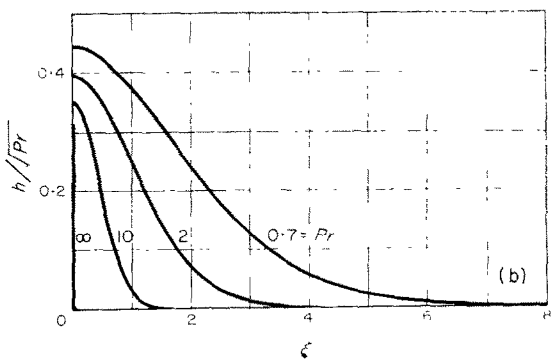

FIG. 2b. Nondimensional temperature distributions above a line heat source, $h/\sqrt{Pr}$.

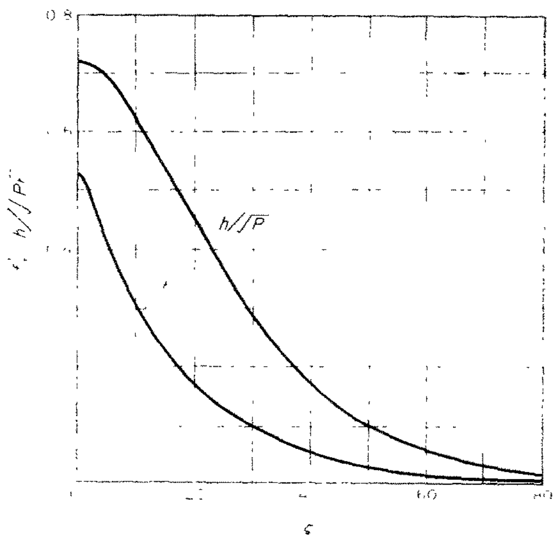

FIG. 3. Nondimensional vertical velocity and temperature distributions above a line heat source for $Pr = 0{\cdot}01$, f' and $h/\sqrt{Pr}$.

and the tables of numerical results are shown in [3]. Besides, Fig. 2a contains the result obtained by Spalding and Cruddace [2] for $Pr \to \infty$. Schuh's result [1] for $Pr = 0{\cdot}7$, seems to coincide well with corresponding curves in Figs. 2.

The accuracy of the calculation is checked, for example, by the relation of conservation of momentum. Then the differences between each terms of (14) are less than $0{\cdot}2$ per cent of either term for the solution of each Prandtl number.

By the way equation (14) is useful to determine the eigenvalue $f'(0)$ for the case where $Pr \to \infty$. This method makes the calculation more easy and accurate than that of Spalding and Cruddace [2].

II. A POINT HEAT SOURCE

In this chapter a column of heated fluid above a point heat source is analysed. Ordinary differential equations, analytical solution for $Pr = 2$, method of numerical calculation and so on are reduced similarly to those in the previous chapter. For the sake of simplicity the results and the procedures of the reduction are listed without explanation, and the correspondence is shown by adding to each equation the same number as in the previous chapter, namely (II,). The cylindrical co-ordinate system in this case is also indicated in Fig. 1.

where y and v signify co-ordinate of horizontal radial direction and velocity component of y direction respectively.

II. 1. *Basic equations*

Basic differential equations and boundary conditions are:

$$\frac{\partial(yu)}{\partial x} + \frac{\partial(yv)}{\partial y} = 0, \qquad \text{(II, 1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta t + \nu\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right), \qquad \text{(II, 2)}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \kappa\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial t}{\partial y}\right), \qquad \text{(II, 3)}$$

$$\left.\begin{aligned} &y = 0; v = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial t}{\partial y} = 0, \\ &y = \infty;\ u = 0, t = 0. \end{aligned}\right\} \qquad \text{(II, 4)}$$

Stream function ψ is introduced as:

$$\frac{1}{y}\frac{\partial\psi}{\partial y} = u, \quad \frac{1}{y}\frac{\partial\psi}{\partial x} = -v. \qquad \text{(II, 5)}$$

Transformation formula from the partial differential equations to ordinary ones are

$$\xi = G_p^{1/4}(y/x), \psi = \nu x f(\xi), t = \Theta_p h(\xi), \qquad \text{(II, 6)}$$

where

$$G_p = \frac{x^3 g\beta\Theta_p}{\nu^2}, \Theta_p = \frac{Q_p}{2\pi c_p \gamma \nu x}, \qquad \text{(II,7), (II, 13)}$$

and Q_p is the rate of heat transfer from a point source (kcal/h),

The ordinary differential equations and the boundary conditions are:

$$\left.\begin{aligned} &\frac{f'''}{\xi} + \frac{(f-1)}{\xi}\left(\frac{f'}{\xi}\right)' + h = 0, \\ &(\xi h')' + Pr(fh)' = 0, \end{aligned}\right\} \text{(II, 8a, b)}$$

$$\left.\begin{aligned} &\xi = 0;\ \frac{f}{\xi} - \frac{f'}{2} = 0, \left(\frac{f'}{\xi}\right)' = 0, h' = 0, \\ &\xi = \infty;\ \frac{f'}{\xi} = 0, h = 0. \end{aligned}\right\} \qquad \text{(II, 9)}$$

The vertical and horizontal velocity components u and v are written as:

$$\frac{ux}{\nu} = G_p^{\frac{1}{2}}\frac{f'}{\xi}, \quad \frac{vy}{\nu} = -\xi\left(\frac{f}{\xi} - \frac{f'}{2}\right). \text{(II, 10)}$$

Conservation of momentum is expressed as:

$$\int_0^\infty \frac{f'^2}{\xi}\,d\xi = \int_0^\infty h\xi\,d\xi. \qquad \text{(II, 14)}$$

In these reductions f and h, the solution of equation (II, 8), are normalized by the following formula:

$$\textstyle\int_0^\infty f'h\,d\xi = 1. \qquad \text{(II, 15)}$$

Schuh's calculation [1] corresponds to that f and h are normalized by $2\pi\int_0^\infty f'h\,d\xi = 1$.

II. 2. *Analytical solution of the ordinary differential equations for* $Pr = 2$

By assuming:

$$\frac{f'^2}{\xi} - h\xi \equiv 0, \qquad \text{(II, 16)}$$

we obtain:

$$\left(f'' - \frac{f'}{\xi} + \frac{ff'}{\xi}\right)' = 0. \qquad \text{(II, 17)}$$

This equation has the same form as the equation given in the problem of a round jet stream [4].

The solution of Equations (II, 8) for $Pr = 2$ is written as:

$$f = \frac{\alpha\xi^2}{1 + (\alpha/4)\xi^2}, \quad h = \frac{4\alpha^2}{[1 + (\alpha/4)\xi^2]^4},$$

(II, 19), (II, 20)

and the nondimensional velocity components are:

$$\left.\begin{aligned} &\frac{\xi}{f'} = \frac{2\alpha}{[1 + (\alpha/4)\xi^2]^2}, \\ &-\xi\left(\frac{f}{\xi} - \frac{f'}{2}\right) = -\frac{\alpha^2\xi^4}{4[(1 + (\alpha/4)\xi^2]^2}, \end{aligned}\right\} \qquad \text{(II, 22)}$$

where

$$\alpha = \sqrt{(5)/4} \doteqdot 0{\cdot}559. \qquad \text{(II, 21)}$$

The solution of equation (II, 8) for $Pr = 1$ are also obtained as:

$$f = \frac{\xi^2}{2(1 + \frac{1}{12}\xi^2)}, \quad h = \frac{2}{3}\frac{1}{(1 + \frac{1}{12}\xi^2)^3},$$

and the nondimensional velocity components are:

$$\frac{f'}{\xi} = \frac{1}{(1 + \frac{1}{12}\xi^2)^2},$$

$$\xi\left(\frac{f}{\xi} - \frac{f'}{2}\right) = \frac{\xi^4}{24(1 + \frac{1}{12}\xi^2)^2}.$$

This reduction is described in the Appendix.

II. 3. *Numerical calculation*

The formulae of normalizing an arbitrary pair of solutions of equations (II, 8) $\bar{f}(\bar{\xi})$ and $\bar{h}(\bar{\xi})$ are :

$$\xi = J^{\frac{1}{4}}\bar{\xi}, \quad f = \bar{f}, \quad h = J^{-1}\bar{h}, \qquad \text{(II, 26)}$$

where $J = \int_0^\infty \bar{f}'\bar{h}\, d\bar{\xi}$.

In Figs. 4a and b and Fig. 5 are shown the nondimensional vertical velocity distribution f'/ξ and the nondimensional temperature distribution h. In these figures, the curves for $Pr = 1$ and 2 show analytical solutions and those for $Pr = 0{\cdot}01$, 0·7 and 10 show numerical solutions by an electronic digital computer. Details of the numerical calculation are shown in [3].

The accuracy of the numerical results, checked by the relation of conservation of momentum, shows that the differences between each term of (II, 14) is less than 1 per cent of either term, for the solution of each Prandtl number. Schuh's result for $Pr = 0{\cdot}7$ seems to have slightly higher values at $\xi = 0$ than those of corresponding curves in Figs. 4a and 4b.

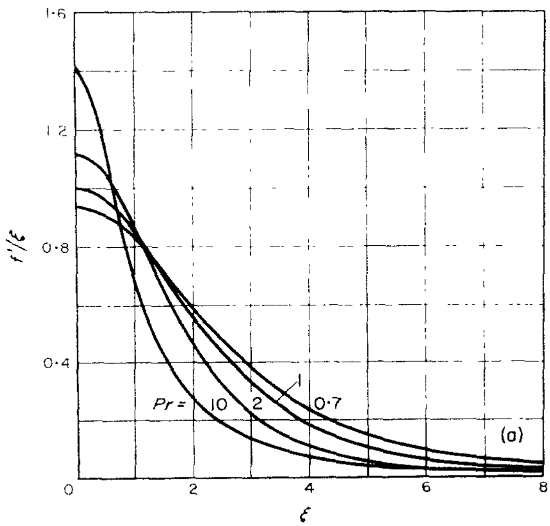

FIG. 4a. Nondimensional vertical velocity distributions above a point heat source, f'/ξ.

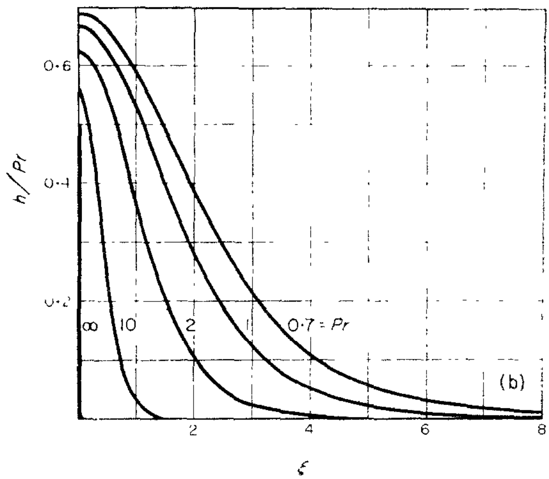

FIG. 4b. Nondimensional temperature distributions above a point heat source, h/Pr.

III. CONCLUSIONS AND DISCUSSIONS

Steady natural convection above a horizontal line heat source and a point heat source are analysed mathematically. The conclusions are as follows:

(i) Dominant nondimensional parameters in this phenomena are, for a line heat source:

$$G_l = \frac{x^3 g\beta\Theta_l}{\nu^2} \text{ and } \Theta_l = \frac{Q_l}{c_p\gamma\nu},$$

and for a point heat source:

$$G_p = \frac{x^3 g\beta\Theta_p}{\nu^2} \text{ and } \Theta_p = \frac{Q_p}{2\pi c_p\gamma\nu x}.$$

(ii) In natural convection the definition of the boundary layer thickness is not yet established generally. When the plume thickness of the heated fluid above a line heat source is defined on the nondimensional $f' - \xi$ plane, for example, the value ξ_1 at $f'(\xi_1) = 0{\cdot}01\, f'(0)$, the real plume thickness y_1 is from (6), (7) and (13):

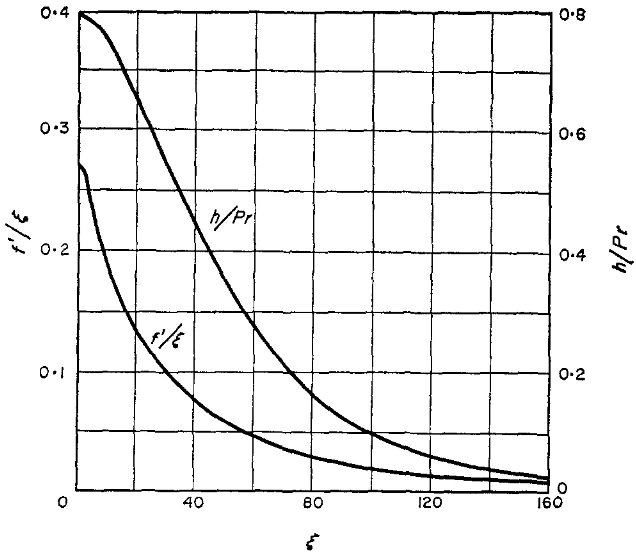

FIG. 5. Nondimensional vertical velocity and temperature distributions above a point heat source for $Pr = 0{\cdot}01$, f'/ξ and h/Pr.

$$y_1 = \left(\frac{c_p \gamma \nu^3}{g\beta Q_l}\right)^{\frac{1}{5}} \xi_1 x^{\frac{2}{5}}.$$

Thus the plume thickens in accordance with the two-fifths power of the distance above the heat source, and thins with one-fifth the power of the heating rate.

Similarly the column radius of the heated fluid above a point heat source y_1 is defined as:

$$y_1 = \left(\frac{2\pi c_p \gamma \nu^3}{g\beta Q_p}\right)^{\frac{1}{4}} \xi_1 x^{\frac{1}{2}}.$$

Thus the column thickens in accordance with the square root of the distance above the heat source, and thins with one-fourth the power of the heating rate.

(iii) Analytical solutions are given for $Pr = 2$; together with (5–7), (13), (15), (19–21) and (26), and the equations of same number in Chapter II. These correspond to the case where following relations exist at any point in the fields of velocity and temperature. From (16) and (II, 16),

$u^2 = \frac{5}{4} x g\beta t$; for a line heat source,

$u^2 = x g\beta t$; for a point heat source.

Especially for a point heat source, an analytical solution is also given for $Pr = 1$ (cf. Section II.2. and Appendix). In this case the relation between vertical velocity component u and temperature t is not so simple as above, that is

$$u^{\frac{3}{2}} = \frac{3}{2}\left(\frac{2\pi g^3 \beta^3 c_p \gamma \nu}{Q_p}\right)^{\frac{1}{4}} x t.$$

(iv) The maximum vertical velocity component $u_{\max}$, the maximum temperature $t_{\max}$ and the rate of flow in vertical direction W are, for a line heat source:

$$u_{l\max} = f_l'(0)\left(\frac{g\beta}{\nu^{\frac{1}{2}} c_p \gamma}\right)^{\frac{2}{5}} x^{\frac{1}{5}} Q_l^{\frac{2}{5}},$$

$$t_{l\max} = h_l(0)\,(g\beta \nu^2 c_p^{\,4} \gamma^4)^{-\frac{1}{5}} x^{-\frac{3}{5}}\, Q_l^{\frac{4}{5}},$$

$$w_l = \int_{-\infty}^{\infty} \gamma u \, dy = 2f(\infty)\left(\frac{\gamma^4 \nu^2 g\beta}{c_p}\right)^{\frac{1}{5}} x^{\frac{3}{5}} Q_l^{\frac{1}{5}}$$

and for a point heat source:

$$u_{p\max} = \left(\frac{f'}{\xi}\right)_0 \left(\frac{g\beta}{2\pi \nu c_p \gamma}\right)^{\frac{1}{2}} Q_p^{\frac{1}{2}},$$

$$t_{p\max} = h_p(0)\,(2\pi \nu c_p \gamma)^{-1} x^{-1}\, Q_p,$$

$$w_p = 2\pi\gamma \int_0^{\infty} u y \, dy = 2\pi f(\infty)\, \gamma \nu x,$$

where the values of $f'(0)$, $h(0)$, . . . , $2\pi f(\infty)$ etc. are shown in Tables 1 and 2.

Table 1. Characteristic values of velocity and temperature distributions above a line heat source

Pr	$f'(0)$	$h(0)$	$h(0)/\sqrt{Pr}$	$2f(\infty)$	δ	$\sqrt{(Pr)}\delta$
0·01	0·531	0·0720	0·720	19·42	16·7	1·67
0·7	0·808	0·373	0·446	4·11	2·63	2·20
2	0·837	0·560	0·396	3·342	1·446	2·045
10	0·875	1·117	0·353	2·90	0·606	1·92
∞	0·933 [1]	∞	0·320	2·69 [1]	0	1·80

Pr; Prandtl number, $f'(0)$; maximum values of nondimensional vertical velocity, $h(0)$; maximum values of nondimensional temperature, $2f(\infty)$; coefficients of the vertical flow rate, δ; temperature layer thickness.

Table 2. Characteristic values of velocity and temperature distributions above a point heat source

Pr	$(f'/\xi)_0$	$h(0)$	$h(0)/Pr$	$2\pi f(\infty)$	δ	$\sqrt{(Pr)}\delta$
0·01	0·269	0·00795	0·759	a. 2970	144	14·4
0·7	0·938	0·481	0·687	47·3	6·11	5·12
1	1·000	0·667	0·667	37·70	4·459	4·459
2	1·117	1·250	0·625	25·13	2·728	3·858
10	1·403	5·61	0·561	16·8	0·98	3·10
∞	—	∞	0·500	-	0	

Pr; Prandtl number, $(f'/\xi)_0$; maximum values of nondimensional vertical velocity, $h(0)$; maximum values of nondimensional temperature, $2\pi f(\infty)$; coefficients of the vertical flow rate, δ; temperature layer thickness.

It is remarkable that the indexes of x and Q are quite different from those of turbulent natural convection [5]. Moreover, it seems to be unreasonable that the rate of flow W_p for a point heat source is independent of the rate of heat transfer. This comes from the fact that the magnitude of the vertical velocity component u and the extension of the velocity field are cancelled out mutually with the variation of the rate of heat transfer. However, when the rate of heat transfer becomes smaller than a limit, the basic equations themselves are not consistent. Of course these regions must be excluded from the problem.

(v) It is recognized in Figs. 2 and Figs. 4 that the variation of the nondimensional vertical velocity distribution f' or f'/ξ with the variation of Prandtl number is relatively small except for $Pr = 0{\cdot}01$. A similar proposition consists as to nondimensional temperature h, when it is arranged by $h/\sqrt{Pr}$ for a line heat source and by h/Pr for a point heat source. From the other point of view, the following relations between the maximum nondimensional temperature h_{max} and Prandtl number are recognized approximately in Table 1 and Table 2:

$$h_{l\,max} \propto \sqrt{Pr}; \text{ for a line heat source,}$$

$$h_{p\,max} \propto Pr; \text{ for a point heat source.}$$

(vi) The temperature layer thickness δ in Table 1 and Table 2 is defined by the following formulae in order to express the extension of the temperature field.

$$2\int_{-\delta/2}^{\delta/2} h\,d\xi = \int_{-\infty}^{\infty} h\,d\xi; \text{ for a line heat source,}$$

$$2\int_{0}^{\delta/2} h\xi\,d\xi = \int_{0}^{\infty} h\xi\,d\xi; \text{ for a point heat source.}$$

From those tables the following relation is recognized approximately

$$\delta \propto \frac{1}{\sqrt{Pr}}.$$

(vii) Fig. 3 and Fig. 5 show that the extension of the velocity field and the temperature field are very large for $Pr = 0{\cdot}01$. It is possible therefore that the simplified equations of motion and energy themselves lose their consistency. Hence, in this case, a question arises whether the solutions hold in the phenomena or not.

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APPENDIX. ANALYTICAL SOLUTION OF EQUATIONS (II, 8) FOR $Pr = 1$

By an analogy from the solution for $Pr = 2$, the solution of (II, 8) may be assumed as:

$$\frac{f'}{\xi} = \frac{A}{[1+(1/B)\xi^2]^2},\ h = \frac{C}{[1+(1/B)\xi^2]^m}, \quad \text{(A)}$$

where A, B, C and m are arbitrary constants. These equations satisfy the conditions (II, 9). Equation (II, 8a), into which equations (A) are substituted, is consistent, when the following equation is satisfied identically,

$$8 + \left(2A - \frac{16}{B}\right)\xi^2 \equiv \frac{CB}{A}\left(1 + \frac{1}{B}\xi^2\right)^{4-m}. \quad \text{(B)}$$

Similarly, the following condition is obtained by (II, 8b):

$$Pr\ AB \equiv 4m. \quad \text{(C)}$$

Equation (B) gives

$$2AB - 16 = 8 \text{ and } CB/A = 8;\ \text{for } m = 3, \quad \text{(D)}$$

or

$$2A - 16/B = 0 \text{ and } CB/A = 8;\ \text{for } m = 4, \quad \text{(E)}$$

and in other values of m (B) does not hold. By solving (C) and (D) or (C) and (E) simultaneously, we obtain:

$$m = 3;\ Pr = 1,\ A = 1,\ B = 12,\ C = 2/3, \quad \text{(F)}$$

$$m = 4;\ Pr = 2,\ AB = 8,\ C = A. \quad \text{(G)}$$

A pair of solution (A) with the coefficients A, B and C expressed by formulae (F) also satisfies the normalizing condition (II, 15). Thus the solution for $Pr = 1$:

$$f = \frac{\xi^2}{2(1 + \frac{1}{12}\xi^2)},\ h = \frac{2}{3}\frac{1}{(1 + \frac{1}{12}\xi^2)^3} \quad \text{(H)}$$

is obtained. The nondimensional velocity component f'/ξ is:

$$\frac{f'}{\xi} = \frac{1}{(1 + \frac{1}{12}\xi^2)^2}. \quad \text{(I)}$$

The relation between f'/ξ and h for $Pr = 1$ is expressed as:

$$\left(\frac{f'}{\xi}\right)^{\frac{3}{2}} = \frac{3}{2}h. \quad \text{(J)}$$

The last equation in (iii) of Chapter III is derived from this equation.

The solution for $m = 4$ in which coefficients A, B and C are determined by condition (II, 15) and (G) coincides with the solution for $Pr = 2$ obtained in Section II. 2.

Résumé—Cet article fait une étude mathématique de la convection naturelle laminaire en régime permanent au-dessus d'une source de chaleur linéaire horizontale et d'une source ponctuelle. Les solutions des fonctions élémentaires sont données pour $Pr = 2$, ainsi que pour $Pr = 1$, dans le cas de l'écoulement au-dessus d'une source ponctuelle.

Les distributions de vitesses et de températures pour $Pr = 0{,}01, 0{,}7$ et 10 sont calculées au calculateur électronique; les differences des distributions de vitesses ou de températures dues à la variation du nombre de Prandtl sont décrites en détail.

Zusammenfassung—Die stationäre laminare freie Konvektion über einer waagerechten linienförmigen und einer punktförmigen Wärmequelle wurde mathematisch analysiert. Für $Pr = 2$ sind die Lösungen der Elementarfunktionen angegeben und für die Strömung über der punktförmigen Wärmequelle auch für $Pr = 1$.

Die Geschwindigkeits- und Temperaturverteilung fur Prandtlzahlen von 0,01, 0,7 und 10 wurde auf einer elektronischen Rechenmaschine ermittelt. Differenzen in den Geschwindigkeits- und Temperaturverteilungen, die von verschiedenen Prandtlzahlen herrühren, sind ausführlich diskutiert.

Аннотация—Дается математический анализ явлений ламинарной стационарной свободной конвекции над линейным горизонтальным и точечным источниками тепла. Приводятся решения в элементарных функциях для $Pr = 2$, а также при обтекании точечного источника для $Pr = 1$.

С помощью электронной счетной машины вычислены распределения скорости и температуры для $Pr = 0{,}01; 0{,}7$ и 10. Дается подробное описание различий в распределении температуры и скорости при разных числах Прандтля.